

# HOMOGENEOUS CONFORMAL AVERAGING OPERATORS ON SEMISIMPLE LIE ALGEBRAS

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**ABSTRACT.** In this note we show a close relation between the following objects: Classical Yang—Baxter equation (CYBE), conformal algebras (also known as vertex Lie algebras), and averaging operators on Lie algebras. It turns out that the singular part of a solution of CYBE (in the operator form) on a Lie algebra  $\mathfrak{g}$  determines an averaging operator on the corresponding current conformal algebra  $\text{Cur } \mathfrak{g}$ . For a finite-dimensional semisimple Lie algebra  $\mathfrak{g}$ , we describe all homogeneous averaging operators on  $\text{Cur } \mathfrak{g}$ . It turns out that all these operators actually define solutions of CYBE with a pole at the origin.

## 1. AVERAGING OPERATORS

Throughout the paper, all linear spaces and algebras considered are over  $\mathbb{C}$ .

**Definition 1.** Suppose  $A$  is a linear algebra (not necessarily associative) with a product  $\cdot : A \otimes A \rightarrow A$ , and let  $T$  be a linear transformation of the space  $A$ . Then  $T$  is an *averaging operator* if

$$T(T(a) \cdot b) = T(a) \cdot T(b) = T(a \cdot T(b)) \quad (1)$$

for all  $a, b \in A$ .

If  $A$  is commutative or anti-commutative then it is enough to check only one of the equations (1).

Averaging operators initially appear in Reynolds' works in turbulence theory and later found many applications in analysis. We refer a reader to the well-written introduction of [6] for more details of the story. Let us state here a couple of simple examples.

**Example 1.** Let  $G$  be a finite group with a linear representation on a space  $V$ . Then the following transformation is an averaging operator on  $A = \text{End } V$ :

$$T : \psi \mapsto \sum_{g \in G} g\psi g^{-1}, \quad \psi \in \text{End } V.$$

**Example 2.** Let  $T_1$  and  $T_2$  be commuting averaging operators on an algebra  $A$ . Then  $T_1 T_2$  is an averaging operator on  $A$ .

**Example 3.** A non-degenerate averaging operator  $T$  on an algebra  $A$  belongs to the centroid of  $A$ . In particular, for finite-dimensional simple algebras  $T = \alpha \text{id}$ ,  $\alpha \in \mathbb{C}$ , for classically semisimple algebras such  $T$  acts as a direct sum of scalar maps on simple summands.

Averaging operators naturally appear in the theory of Leibniz algebras, the most popular and well-studied noncommutative analogues of Lie algebras. Namely, if  $\mathfrak{g}$  is a Lie algebra with a product  $[\cdot, \cdot]$  equipped with an averaging operator  $T$  then the same space relative to new binary operation

$$\{a, b\}_T = [T(a), b], \quad a, b \in \mathfrak{g},$$

satisfies the Jacobi identity  $\{x, \{y, z\}_T\}_T - \{y, \{x, z\}_T\}_T = \{\{x, y\}_T, z\}_T$ , i.e.,  $\mathfrak{g}_T = (\mathfrak{g}, \{\cdot, \cdot\}_T)$  is a Leibniz algebra. Note that  $\text{Ker } T$  is an ideal of  $\mathfrak{g}_T$ , and  $\mathfrak{g}_T / \text{Ker } T$  is a Lie algebra.

Moreover, every Leibniz algebra may be embedded into an algebra of the form  $\mathfrak{g}_T$  for an appropriate Lie algebra  $\mathfrak{g}$  with an averaging operator  $T$ . This is a general statement for di-algebras, see [5].

## 2. CONFORMAL ALGEBRAS

Conformal algebras were introduced by V. Kac as “Lie analogues” of vertex algebras. Namely, the commutator of two chiral fields in 2-dimensional conformal field theory as proposed by [3] may be expressed in terms of coefficients taken from singular part of their operator product expansion (OPE) [7]. This naturally leads to the following

**Definition 2.** A linear space  $C$  equipped with a linear operator  $\partial : C \rightarrow C$  and with a family of bilinear “products”  $[\cdot]_{(n)} \cdot$ , where  $n$  ranges over the set  $\mathbb{Z}_+$  of nonnegative integers, is called *conformal algebra* if for every  $a, b \in C$

(C1) There exists  $N = N(a, b)$  such that  $[a]_{(n)} b = 0$  for  $n \geq N$ ;

(C2)  $[\partial a]_{(n)} b = -n[a]_{(n-1)} b$ ;

(C3)  $[a]_{(n)} \partial b = \partial([a]_{(n)} b) + n[a]_{(n-1)} b$ .

If, in addition,

$$[a]_{(n)} b = \sum_{s \geq 0} (-1)^{n+s+1} \frac{1}{s!} \partial^s ([b]_{(n+s)} a) \quad (2)$$

and

$$[a]_{(n)} [b]_{(m)} c - [b]_{(m)} [a]_{(n)} c = \sum_{s \geq 0} \binom{n}{s} [[a]_{(n-s)} b]_{(m+s)} c \quad (3)$$

for all  $a, b, c \in C$ ,  $n, m \in \mathbb{Z}_+$  then  $C$  is said to be a Lie conformal algebra (also known as *Lie vertex algebra* [4]).

The following example of a conformal algebra, though simplest possible, is essential for our needs.

**Example 4.** Suppose  $\mathfrak{g}$  is a Lie algebra. Then  $C = \mathbb{C}[\partial] \otimes \mathfrak{g}$  equipped with

$$(1 \otimes a)_{(n)} (1 \otimes b) = \delta_{n,0} \otimes [a, b], \quad a, b \in \mathfrak{g}$$

(one may use (C2) and (C3) to expand these operations to the entire  $C$ ) turns into a Lie conformal algebra denoted by  $\text{Cur } \mathfrak{g}$  (current conformal algebra).

Suppose  $C$  is a Lie conformal algebra,  $T : C \rightarrow C$  is a  $\mathbb{C}[\partial]$ -linear map such that  $T([T(a)_{(n)} b]) = [T(a)_{(n)} T(b)]$ ,  $a, b \in C$ ,  $n \in \mathbb{Z}_+$ . This is a close analogue of an ordinary averaging operator in the class of conformal algebras. Then the same module  $C$  equipped with new family of operations  $\{a_{(n)} b\}_T = [T(a)_{(n)} b]$  is a conformal algebra  $C_T$  satisfying (3), i.e., it may be called a conformal Leibniz algebra. As in the case of ordinary Leibniz algebras,  $C_T / \text{Ker } T$  is a Lie conformal algebra.

### 3. SINGULAR PART OF CYBE SOLUTION

Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra. Suppose  $X$  is a  $(\mathfrak{g} \otimes \mathfrak{g})$ -valued function of a complex variable  $u$  which is meromorphic at  $u = 0$ , i.e.,  $X(u)$  is presented by a Laurent series in a neighborhood  $\mathcal{D} \subset \mathbb{C}$  of the origin. As usual, if  $X = \sum_i x'_i \otimes x''_i \in \mathfrak{g} \otimes \mathfrak{g}$  then  $X^{12}$  stands for  $\sum_i x'_i \otimes x''_i \otimes 1 \in U(\mathfrak{g})^{\otimes 3}$ , and the same convention determines  $X^{13}$ ,  $X^{23}$ .

The classical Yang—Baxter equation (CYBE) is the functional equation

$$[X^{12}(u), X^{13}(u+v)] + [X^{12}(u), X^{23}(v)] + [X^{13}(u+v), X^{23}(v)] = 0. \quad (4)$$

A. Belavin and V. Drinfeld in their famous paper [2] classified non-degenerate solutions. Solutions of CYBE are of great interest for pure algebra since they are related to quantizations of Lie bialgebras, see, e.g., [8] and references therein. Constant solutions of CYBE and its generalizations are in good connection with Rota—Baxter operators and their generalizations [1].

If  $\mathfrak{g}$  is a semisimple algebra then its Killing form  $\langle \cdot, \cdot \rangle$  is non-degenerate, and one may identify  $\mathfrak{g} \otimes \mathfrak{g}$  with  $\text{End } \mathfrak{g}$  by the natural rule

$$a \otimes b \mapsto \varphi_{a \otimes b}, \quad \varphi_{a \otimes b}(x) = \langle a, x \rangle b,$$

for  $a, b, x \in \mathfrak{g}$ . For example, the Casimir tensor  $\Omega$  corresponds to the identity map. It is easy to see that  $\varphi_{b \otimes a} = \varphi_{a \otimes b}^*$ , where  $*$  stands for the conjugation in  $\text{End } \mathfrak{g}$  relative to the Killing form.

Therefore, a meromorphic tensor-valued function  $X$  corresponds to a meromorphic operator-valued function  $P_u = \varphi_{X(u)}$ ,  $u \in \mathcal{D}$ . Straightforward computation shows that (4) is equivalent to the following operator equation:

$$P_{u+v}([x, P_u^*(y)]) - P_v([P_u(x), y]) + [P_{u+v}(x), P_v(y)] = 0 \quad (5)$$

for  $x, y \in \mathfrak{g}$ . In particular, if  $P_u = R$  is a skew-symmetric constant then  $R$  satisfies

$$[R(x), R(y)] = R([x, R(y)]) + R([x, R(y)]),$$

i.e., is a Rota—Baxter operator.

**Example 5.** Suppose  $T$  is a symmetric (relative to the Killing form) averaging operator on a semisimple Lie algebra  $\mathfrak{g}$ . Then

$$P_u(a) = \frac{1}{u} T(a), \quad a \in \mathfrak{g},$$

is a solution of the operator CYBE (5)

It is enough for  $T$  to have the following symmetry property:

$$T([T^*(x), y] - [T(x), y]) = 0$$

for every  $x, y \in \mathfrak{g}$ .

It was established in [2] for a simple Lie algebra  $\mathfrak{g}$  that if  $P_u$  is a non-degenerate solution of CYBE (that is,  $\det P_u \neq 0$  at some point  $u \in \mathcal{D}$ ) then the singular part of  $P_u$  is of the form  $\frac{\lambda \text{id}}{u}$ ,  $\lambda \in \mathbb{C}$ .

In general, a solution of (5) may have a more complicated singular part, which may be presented by its generating function—a polynomial in a formal variable  $\lambda$ :

$$T_\lambda = \text{Res}_{u=0} P_u \exp(\lambda u) \in (\text{End } \mathfrak{g})[\lambda]. \quad (6)$$

Let us deduce an equation on  $T_\lambda$ .

**Theorem 1.** *Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra and let  $P_u$  be a meromorphic at  $u = 0$  solution of CYBE (5). Then  $T_\lambda$  given by (6) satisfies*

$$T_{\lambda+\mu}([T_\lambda(x), y]) = [T_\lambda(x), T_\mu(y)] \in \mathfrak{g}[\lambda, \mu] \quad (7)$$

for all  $x, y \in \mathfrak{g}$ .

*Proof.* Let us multiply (5) by  $\exp((\lambda + \mu)v) \exp(\lambda u)$  and integrate the expression obtained:

$$\begin{aligned} \oint_{|u|=r} \oint_{|v|=d} \exp((\lambda + \mu)v) \exp(\lambda u) (P_v([P_u(x), y]) - P_{u+v}([x, P_u^*(y)])) dv du \\ = \oint_{|u|=r} \oint_{|v|=d} \exp(\lambda(u + v)) \exp(\mu v) [P_{u+v}(x), P_v(y)] dv du \end{aligned} \quad (8)$$

where  $d < r$  and  $d$  is small enough for the entire circle  $|u| \leq 2r$  to lie in  $\mathcal{D}$ . The first summand of the left-hand side of (8) obviously provides  $T_{\lambda+\mu}([T_\lambda(x), y])$ , the second one is equal to zero since for every fixed  $u$ ,  $|u| = r$ ,  $P_{u+v}([x, P_u^*(y)])$  considered as a function on  $v$  has no poles in  $|v| \leq d < r$ . The right-hand side of (8) can be calculated by substitution:

$$\begin{aligned} \oint_{|u|=r} \oint_{|v|=d} \exp((\lambda + \mu)v) \exp(\lambda u) [P_{u+v}(x), P_v(y)] dv du \\ = \oint_{|v|=d} \oint_{|u|=r} [\exp(\lambda(u + v)) P_{u+v}(x), \exp(\mu v) P_v(y)] du dv \\ = \oint_{|v|=d} \oint_{\substack{w=v+u \\ |u|=r}} [\exp(\lambda w) P_w(x), \exp(\mu v) P_v(y)] dw dv, \end{aligned}$$

which gives  $[T_\lambda(x), T_\mu(y)]$  since  $w = 0$  is the only singular point in the interior of the curve  $w = v + u$ ,  $|u| = r$ . Therefore, (7) holds.  $\square$

**Definition 3.** Let us call an operator  $T_\lambda : \mathfrak{g} \rightarrow \mathfrak{g}[\lambda]$  satisfying (7) by *conformal averaging operator* on  $\mathfrak{g}$ .

The explanation of the term “conformal” comes from the following observation. Suppose  $C = \text{Cur } \mathfrak{g}$  is the current conformal algebra over  $\mathfrak{g}$ , and let  $T_\lambda$  be a conformal averaging operator on  $\mathfrak{g}$ . It turns out that  $T_\lambda$  is a form of an averaging operator on the conformal algebra  $\text{Cur } \mathfrak{g}$ .

**Proposition 1.** Assume  $T_\lambda$  is a conformal averaging operator on  $\mathfrak{g}$ ,

$$T_\lambda(a) = \sum_{n \geq 0} \lambda^{(n)} T_n(a), \quad a \in \mathfrak{g},$$

where  $\lambda^{(n)}$  stand for  $\lambda^n/n!$ ,  $T_n$  are linear transformations of  $\mathfrak{g}$ . Consider  $\mathbb{k}[\partial]$ -linear map  $T : \text{Cur } \mathfrak{g} \rightarrow \mathfrak{g}$ , given by

$$T(1 \otimes a) = T_{-\partial}(a) = \sum_{n \geq 0} (-\partial)^{(n)} \otimes T_n(a).$$

Then

$$T([T(x) \text{ } (n) \text{ } y]) = [T(x) \text{ } (n) \text{ } T(y)]$$

for all  $x, y \in \text{Cur } \mathfrak{g}$ ,  $n \in \mathbb{Z}_+$ .

*Proof.* It is enough to prove the statement for  $x, y \in \mathfrak{g} \simeq 1 \otimes \mathfrak{g}$ . Compare coefficients at  $\lambda^{(n)} \mu^{(m)}$  in (7):

$$[T_n(x), T_m(y)] = \sum_{t \geq 0} \binom{n}{t} T_{m+t}([T_{n-t}(x), y]). \quad (9)$$

Conversely (replace  $\mu$  with  $\mu - \lambda$ ),

$$T_m([T_n(x), y]) = \sum_{s \geq 0} (-1)^s \binom{n}{s} [T_{n-s}(x), T_{m+s}(y)]. \quad (10)$$

It remains to calculate left- and right-hand sides of the desired relation in  $\text{Cur } \mathfrak{g}$ :

$$\begin{aligned} T([T(x) \text{ } (n) \text{ } y]) &= \sum_{m \geq 0} (-\partial)^{(m)} \otimes T_m([T_n(x), y]), \\ [T(x) \text{ } (n) \text{ } T(y)] &= \sum_{k, s \geq 0} \binom{n}{k} [T_k(x) \text{ } (n-k) \text{ } (-\partial)^{(s)} T_s(y)] \\ &= \sum_{k+s=n} (-1)^s \binom{n}{k} [T_k(x), T_s(y)]. \end{aligned}$$

□

Therefore, every solution of CYBE (5) gives rise to a conformal averaging operator  $T_\lambda$  on  $\mathfrak{g}$  and thus induces a Leibniz conformal algebra structure on  $L = \mathbb{C}[\partial] \otimes \mathfrak{g}$  which is  $(\text{Cur } \mathfrak{g})_T$ . Then  $(\text{Cur } \mathfrak{g})_T / \text{Ker } T$  is a Lie conformal algebra induced by the initial solution of CYBE. The purpose of the next section is to describe such Lie conformal algebras.

## 4. DESCRIPTION OF HOMOGENEOUS CONFORMAL AVERAGING OPERATORS

Let  $T_\lambda$  be a conformal averaging operator on a finite-dimensional Lie algebra  $\mathfrak{g}$ . For a subspace  $B$  of  $\mathfrak{g}$  denote by  $T_*(B)$  the linear span of  $T_\alpha(b)$ ,  $b \in B$ ,  $\alpha \in \mathbb{C}$ . Note that if  $B$  satisfies the condition  $[T_*(B), B] \subseteq B$  then  $T_*(B)$  is a subalgebra of  $\mathfrak{g}$  by (9). In particular,  $T_*(\mathfrak{g})$ ,  $T_*(T_*(\mathfrak{g}))$  are subalgebras of  $\mathfrak{g}$ . Let us call  $T_\lambda$  *non-degenerate* if  $T_*(\mathfrak{g}) = \mathfrak{g}$ . If  $\mathfrak{g}$  is semisimple and  $T_*(\mathfrak{g})$  contains Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  then  $T_\lambda$  is called *homogeneous*.

**Lemma 1.** *Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra. Then every non-degenerate conformal averaging operator on  $\mathfrak{g}$  is just an ordinary averaging operator.*

*Proof.* Suppose  $T_\lambda = T_0 + \lambda T_1 + \dots + \lambda^{(N)} T_N$ . It follows from (9) that  $T_N(\mathfrak{g})$  is an ideal of  $T_*(\mathfrak{g})$ , moreover, if  $N > 0$  then this ideal is abelian. Hence,  $N = 0$ , and  $T_\lambda(a) = T_0(a)$  for every  $a \in \mathfrak{g}$ , where  $T_0$  is a non-degenerate averaging operator on  $\mathfrak{g}$ . All such operators are given by 3.  $\square$

**Lemma 2.** *Let  $\mathfrak{g}$  be a finite-dimensional reductive Lie algebra,  $\mathfrak{g} = \mathfrak{g}_0 \oplus Z$ , where  $\mathfrak{g}_0$  is semisimple,  $Z$  is the center of  $\mathfrak{g}$ . Suppose  $T$  is a conformal averaging operator on  $\mathfrak{g}$  such that  $\mathfrak{g}_0 \subseteq T_*(\mathfrak{g})$ . Then  $T_*(\mathfrak{g}_0) = \mathfrak{g}_0$ ,  $T_*(Z) \subseteq Z$ .*

As a corollary,  $T_\lambda$  is a non-degenerate conformal averaging operator on  $\mathfrak{g}_0$  described by Lemma 1. Thus, an ordinary averaging operator on  $\mathfrak{g}_0$  together with an arbitrary map  $Z \rightarrow Z[\lambda]$ , defines a conformal averaging operator  $T_\lambda$  on  $\mathfrak{g}$ .

*Proof.* Relation (9) implies that  $T_m(z)$  commutes with  $\mathfrak{g}_0 \subseteq T_*(\mathfrak{g})$  for  $z \in Z$ . Hence, the induced map  $\bar{T}_\lambda$  is a non-degenerate conformal averaging operator  $T_0$  on  $\mathfrak{g}_0 \simeq \mathfrak{g}/Z$ . By Lemma 1  $T_0 : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ . Therefore,

$$T_\lambda(a) = T_0(a) + \zeta_\lambda(a), \quad a \in \mathfrak{g}_0,$$

where  $\zeta_\lambda : \mathfrak{g}_0 \rightarrow Z[\lambda]$ .

It follows from (7) that  $T_{\lambda+\mu}([T_0(a), b]) = [T_0(a), T_\mu(b)]$ , so  $\zeta_{\lambda+\mu}([T_0(a), b]) = 0$  for all  $a, b \in \mathfrak{g}_0$ . Hence,  $\zeta_\lambda = 0$ .  $\square$

If  $\mathfrak{g}$  is finite-dimensional then  $T_\lambda$  is defined by a finite family of linear operators  $T_n : \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $n = 0, \dots, N$ , as in Proposition 1.

Consider the following chain of subspaces in  $\mathfrak{g}$ :

$$T_{(0)}(\mathfrak{g}) \supseteq T_{(1)}(\mathfrak{g}) \supseteq \dots \supseteq T_{(N)}(\mathfrak{g}),$$

where

$$T_{(n)}(\mathfrak{g}) = T_n(\mathfrak{g}) + \dots + T_N(\mathfrak{g}), \quad n = 0, 1, \dots, N.$$

Suppose  $T_{(N+1)}(\mathfrak{g}) = \{0\}$ . Relations (9) imply that  $T_{(n)}(\mathfrak{g})$  is an ideal of  $T_*(\mathfrak{g})$ . Moreover, for every  $n = 0, 1, \dots, N$  the induced map  $\bar{T}_n : \mathfrak{g} \rightarrow T_*(\mathfrak{g})/T_{(n+1)}(\mathfrak{g})$  given by  $\bar{T}_n(a) = T_n(a) + T_{(n+1)}(\mathfrak{g})$  is a homomorphism of  $T_*(\mathfrak{g})$ -modules:

$$[T_k(x), T_n(a)] - T_n([T_k(x), a]) \in T_{(n+1)}(\mathfrak{g}), \quad x, a \in \mathfrak{g}. \quad (11)$$

**Theorem 2.** *Let  $T_\lambda$  be a homogeneous conformal averaging operator on a semisimple finite-dimensional Lie algebra  $\mathfrak{g}$ . Then  $T_*(\mathfrak{g}) = \mathfrak{g}_0 \oplus Z$  is a reductive Lie algebra and  $\mathfrak{g}_0 \subseteq T_*(T_*(\mathfrak{g}))$ .*

*Proof.* Recall that  $T_\lambda$  is homogeneous if  $T_*(\mathfrak{g})$  contains its Cartan subalgebra  $\mathfrak{h}$ . Suppose  $\Delta \subseteq \mathfrak{h}^*$  is the root system of  $\mathfrak{g}$  and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathbb{C}x_\alpha$$

is the root decomposition of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ . Then  $T_*(\mathfrak{g})$  is a homogeneous subalgebra relative to this root grading. Denote

$$\Delta' = \{\alpha \in \Delta \mid x_\alpha \in T_*(\mathfrak{g})\}.$$

To prove reductivity of  $T_*(\mathfrak{g})$  it is enough to show that  $\Delta'$  is symmetric, i.e., for every  $\alpha \in \Delta'$  we have  $-\alpha \in \Delta'$ . If so,  $\Delta'$  satisfies all necessary axioms of a root system, and the subalgebra  $\mathfrak{g}_0 \subseteq T_*(\mathfrak{g})$  generated by  $\{x_\alpha \mid \alpha \in \Delta'\}$  is a semisimple Lie algebra with Cartan subalgebra  $\mathfrak{h}_0 = \text{Span}\{h_\alpha = [x_\alpha, x_{-\alpha}] \mid \alpha \in \Delta'\}$ . Finally,  $\mathfrak{h} = \mathfrak{h}_0^\perp$  relative to the Killing form, and  $[\mathfrak{h}_0^\perp, x_\alpha] = 0$  for all  $\alpha \in \Delta'$ . Hence,  $T_*(\mathfrak{g}) = \mathfrak{g}_0 \oplus \mathfrak{h}_0^\perp$  is a reductive Lie algebra.

Let us show symmetry of  $\Delta'$ . Assume  $x_\alpha \in T_*(\mathfrak{g})$ . Choose the maximal  $n \in \{0, \dots, N\}$  such that  $x_\alpha \in T_{(n)}(\mathfrak{g})$ :

$$x_\alpha = T_n(y) + b, \quad y \in \mathfrak{g}, \quad b \in T_{(n+1)}(\mathfrak{g}).$$

Note that (11) implies

$$T_n(x_\gamma) - \xi_{n,\gamma}x_\gamma \in T_{(n+1)}(\mathfrak{g}),$$

for  $\xi_{n,\gamma} \in \mathbb{C}$ ,  $\gamma \in \Delta$ , and  $T_n(h) \in \mathfrak{h} + T_{(n+1)}(\mathfrak{g})$  for  $h \in \mathfrak{h}$  since  $T_\lambda$  is homogeneous. Suppose  $y = h + \zeta_\alpha x_\alpha + \sum_{\beta \in \Delta \setminus \{\alpha\}} \zeta_\beta x_\beta$  is the root decomposition of  $y$ . Then  $T_{(n+1)}(\mathfrak{g}) \ni x_\alpha - T_n(y) = (1 - \xi_{n,\alpha}\zeta_\alpha)x_\alpha - T_n(h) - \sum_{\beta \in \Delta \setminus \{\alpha\}} \xi_{n,\beta}\zeta_\beta x_\beta$ . Since  $T_{(n+1)}(\mathfrak{g})$  is a homogeneous subalgebra relative to the root grading and  $x_\alpha \notin T_{(n+1)}(\mathfrak{g})$ , we have  $1 - \xi_{n,\alpha}\zeta_\alpha = 0$ . Thus,  $\xi_{n,\alpha} \neq 0$ .

Assume  $\xi_{n,-\alpha} = 0$ :

$$T_n(x_{-\alpha}) \in T_{(n+1)}(\mathfrak{g}).$$

Then  $[x_\alpha, T_n(x_{-\alpha})] \in T_{(n+1)}(\mathfrak{g})$ . On the other hand, (11) implies

$$[x_\alpha, T_n(x_{-\alpha})] - T_n([x_\alpha, x_{-\alpha}]) \in T_{(n+1)}(\mathfrak{g}),$$

and thus  $T_n(h_\alpha) = T_n([x_\alpha, x_{-\alpha}]) \in T_{(n+1)}(\mathfrak{g})$ . However,

$$[x_\alpha, T_n(h_\alpha)] - T_n([x_\alpha, h_\alpha]) \in T_{(n+1)}(\mathfrak{g})$$

provides contradiction to  $x_\alpha \notin T_{(n+1)}(\mathfrak{g})$  since  $[h_\alpha, x_\alpha] = (\alpha, \alpha)x_\alpha \neq 0$ .

Therefore,  $T_n(x_{-\alpha}) \in \xi_{n,-\alpha}x_{-\alpha} + T_{(n+1)}(\mathfrak{g})$ ,  $\xi_{n,-\alpha} \neq 0$ , and thus  $x_{-\alpha} \in T_*(\mathfrak{g})$ .

We have shown that if  $x_\alpha \in T_*(\mathfrak{g})$  then there exists  $n$  such that  $T_n(x_\alpha) = \xi_{n,\alpha}x_\alpha + T_{(n+1)}(\mathfrak{g})$ ,  $x_\alpha \notin T_{(n+1)}(\mathfrak{g})$ . Therefore, every such  $x_\alpha$  belongs to  $T_*(T_*(\mathfrak{g}))$ . This implies  $\mathfrak{g}_0 \subseteq T_*(T_*(\mathfrak{g}))$ .  $\square$

Theorem 2 allows us to find explicit description of homogeneous conformal averaging operators on a finite-dimensional semisimple Lie algebra  $\mathfrak{g}$  (i.e., on  $\text{Cur } \mathfrak{g}$ ).

**Corollary 1.** *Under the conditions of Theorem 2,  $\mathfrak{g} = T_*(\mathfrak{g}) \oplus \text{Ker } T_\lambda$ .*

*Proof.* By Theorem 2,  $T_\lambda$  is a non-degenerate conformal averaging operator on the reductive Lie algebra  $T_*(\mathfrak{g})$ : There exists root subsystem  $\Delta' \subseteq \Delta$  such that  $T_*(\mathfrak{g})$  is generated by  $\mathfrak{h}$  and  $x_\alpha$ ,  $\alpha \in \Delta'$ . Lemma 2 describes how  $T_\lambda$  acts on  $T_*(\mathfrak{g})$ , in particular,  $T_*(\mathfrak{h}) \subseteq \mathfrak{h}$ .

It remains to show that  $T_\lambda(x_\beta) = 0$  for all  $\beta \in \Delta \setminus \Delta'$ . Assume there exists such a root  $\beta$  that  $T_\lambda(x_\beta) \neq 0$ ,  $x_\beta \notin T_*(\mathfrak{g})$ . Let us choose maximal  $n \geq 0$  such that  $T_n(\beta) \neq 0$ . Then (11) and  $T_k(h) \in \mathfrak{h}$  (for all  $k \geq 0$ ,  $h \in \mathfrak{h}$ ) imply  $[h, T_n(x_\beta)] = \beta(h)T_n(x_\beta)$ , i.e.,  $T_n(x_\beta) \in \mathbb{C}x_\beta$ . As  $x_\beta \notin T_*(\mathfrak{g})$ , we have  $T_n(x_\beta) = 0$ , a contradiction.  $\square$

Finally, we may describe all homogeneous conformal averaging operators on a finite-dimensional semisimple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h}$  and root system  $\Delta$  as follows.

For a root subsystem  $\Delta'$  of  $\Delta$ ,

$$T_\lambda(x_\alpha) = \xi_\alpha x_\alpha, \quad T_\lambda(h_\alpha) = \xi_\alpha h_\alpha,$$

where  $\alpha \in \Delta'$ ,  $h_\alpha = [x_\alpha, x_{-\alpha}]$ . Note that  $\xi_\alpha \in \mathbb{C}$  are nonzero constants that depend on the semisimple decomposition of the semisimple Lie algebra  $\mathfrak{g}_0$  generated by  $x_\alpha$ ,  $\alpha \in \Delta'$ . (In particular,  $\xi_{-\alpha} = \xi_\alpha$ .)

The subalgebra  $\mathfrak{h}_0^\perp = \{h \in \mathfrak{h} \mid \alpha(h) = 0, \alpha \in \Delta'\}$  is invariant with respect to  $T_\lambda$ , and there are no restrictions on functions  $T_n|_{\mathfrak{h}_0^\perp}$  except for their finite number ( $0 \leq n \leq N$ ).

Other root spaces  $\mathbb{C}x_\beta$ ,  $\beta \in \Delta \setminus \Delta'$ , belong to the kernel of  $T_\lambda$ .

Straightforward computation shows

$$[\mathfrak{g}, \mathfrak{h}_0^\perp] \subseteq \text{Ker } T_\lambda, \quad [\mathfrak{h}_0^\perp, T_*(\mathfrak{g})] = 0 \quad (12)$$

since  $[x_\alpha, h] = 0$  for all  $\alpha \in \Delta'$ ,  $h \in \mathfrak{h}_0^\perp$ .

Therefore, we have

**Corollary 2.** *For a homogeneous conformal averaging operator  $T_\lambda$  on  $\mathfrak{g}$  the induced Lie conformal algebra structure  $(\text{Cur } \mathfrak{g})_T / \text{Ker } T$  is given by split null extension  $(\text{Cur } \mathfrak{g}_0) \oplus (\text{Cur } \mathfrak{h}_0^\perp)$ .*

It is worth mentioning that, in general, a conformal averaging operator  $T_\lambda$  may not be a singular part of a solution of CYBE. However, it turns out that all homogeneous conformal averaging operators actually come from some meromorphic solutions of CYBE.

**Corollary 3.** *Let  $T_\lambda$  be a homogeneous conformal averaging operator on a finite-dimensional semisimple Lie algebra  $\mathfrak{g}$ . Then the operator-valued meromorphic function  $P : \mathbb{C} \setminus \{0\} \rightarrow \text{End } \mathfrak{g}$  given by*

$$P_u(a) = u^{-1}T_{u^{-1}}(a), \quad a \in \mathfrak{g},$$

*is a solution of the classical Yang—Baxter equation (5).*



*Proof.* It is enough to check whether (5) holds if either of  $x$  or  $y$  belong to  $\mathfrak{h}_0^\perp$ . First, it is easy to verify that

$$T_\lambda^* = \sum_{n=0}^N \lambda^{(n)} T_n^*,$$

where  $T_n^*$  is the conjugate of  $T_n$  relative to the Killing form, is also a homogeneous conformal averaging operator with the same root subsystem  $\Delta'$  as  $T_\lambda$ .

Next, if  $x \in \mathfrak{h}_0^\perp$  then  $P_u(x) \in h_0^\perp$  for every  $u \in \mathbb{C} \setminus \{0\}$  and thus all three summands of (5) are equal to zero due to (12).

Finally, for  $y \in \mathfrak{h}_0^\perp$  equation (5) holds by the same reason.  $\square$

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